More Properties for NP-complete Minesweeper Graphs

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Abstract

This article deals with the question whether minesweeper graphs with bounded vertex degrees $d \leq 3$ are NP-complete. The answer to this question—stated to be open in [3]—will be *positive* which gives a clear border between simple polynomially solvable and NP hard instances with regard to the vertex degrees: Bounding those by 2 has been shown to cause simple graphs while allowing vertices to have 3 neighbors leads to hard instances in general.

Additionally we discuss more possible properties for NP-complete minesweeper graphs and find a simple way to reduce some classes of graphs to 3SAT. The first section also gives a short overview of Minesweeper's generalization on graphs.

Minesweeper (MW) is the well-known puzzle-game coming with the Microsoft Windows operating system. The player sees a rectangular board of fields with hidden mines. He must visit/open all secure fields and only knows the number of mines surrounding a visited field. He may mark detected mines with a flag, but must not visit those fields.

1 Minesweeper on Graphs

The simple model of MW which consists of fields $\mathcal{F} = M \times N$ and a neighborhood function $\mathcal{N}$—derived from the maximum metric—can be generalized in a very natural way: Let $\mathcal{F}$ be an arbitrary set of fields and $\mathcal{N} : \mathcal{F} \times \mathcal{F} \to \mathbf{N}$ a *weighted* neighborhood function, we interpret $\langle \mathcal{F}, \mathcal{N} \rangle$ as the generalized adjacency-matrix of a graph[1].

[1] We are interested in a graph's vertices and the number of edges between two of them and assume a fitting isomorphism to be given.

Following the standard model, let $\mathcal{M}$ be the set of mines—i.e. the subset of fields containing a mine—we name $\mu \equiv \chi_{\mathcal{M}}$ its characteristic function *mines indicator* and define *surrounding danger* σ and *explosivity* ϵ as follows:

$$\begin{array}{rrcl} \sigma: & \mathcal{F} & \rightarrow & \mathbf{N} \\ & x & \mapsto & \sum_{y\in\mathcal{F}} \mathcal{N}(x,y)\,\mu(y) \end{array} \tag{1}$$

$$\begin{array}{rrcl} \epsilon: & \mathcal{F} & \rightarrow & \mathbf{N}^{\infty} \\ & x & \mapsto & \begin{cases} \sigma(x) & if\ \mu(x)=0 \\ \infty & else \end{cases} \end{array} \tag{2}$$

The equivalence $\epsilon \equiv \infty$ on the mines $\mathcal{M}$ models the fact that those fields provide no information on their neighbors.

The definition of a *configuration* helps us to hide information of unvisited/closed fields: For a board $\langle \mathcal{F}, \mathcal{N} \rangle$ the tuple $\langle \mathcal{F}, \mathcal{N}, \mathcal{D}, \kappa \rangle$ with $\mathcal{D} \subseteq \mathcal{F}$ the visited fields and $\kappa : \mathcal{D} \rightarrow \mathbf{N}^{\infty}$ the values displayed describes a snapshot we can imagine. Such a configuration need not be derived from a real situation, there is no a priori connection between κ and ϵ. This remark directly leads to the *Consistency Problem for Minesweeper on Graphs* (MWG):

Problem 1 (MWG) *Given a configuration $\langle \mathcal{F}, \mathcal{N}, \mathcal{D}, \kappa \rangle$, is there a set of mines $\mathcal{M} \subseteq \mathcal{F}$ such that $\kappa \equiv \epsilon|_{\mathcal{D}}$ the values displayed show the explosivity on the visited fields?*

For examining the complexity of MWG we need to describe instances of the problem on a computer. A configuration consists of the adjacency-matrix $\mathcal{N}$ and the partial function κ. The structure of the matrix is 3-dimensional because it needs to assign integers to pairs (of fields), its size is $|\mathcal{F}|$-by-$|\mathcal{F}|$-by-$\max \mathcal{N}$. Similarly we find a 2-dimensional structure for the latter with size $|\mathcal{F}|$-by-$\max \kappa$. The ways to handle such structures are well-known.

We show that MWG is in the complexity class NP which contains all problems with a polynomial-time computable *checking relation* (see [1] for a proper definition). Finding a complete search-space for polynomial-time checkable witnesses whose lengths are bounded by a polynomial in the length of the instance proofs membership of NP. Of course μ the mine indicator provides such witnesses:

- The search-space contains all functions $\mu : \mathcal{F} \rightarrow \{0, 1\}$. They can be coded as string of linear length $|\mathcal{F}|$ over a finite (binary) alphabet.

- The runtime of the computation of the explosivity ϵ defined by equations (1) and (2) is bounded by a polynomial as well as the comparison to the displayed values κ.

- Such a witness exists iff the given instance is consistent.

An algorithm on a 3-dimensional Turing machine for solving MWG can be found in [3]. Anyway the witness can be coded within linear bounds of the input length, therefore a *context-sensitive* grammar[2] exists for a language which describes exactly the consistent MWG instances.

2 Reductions

In order to define complexity classes, reductions between problems have been established. Informally we can describe them as a relation $A \leq B$ (say problem A reduces to problem B) which holds iff one can solve problem A given a solving algorithm for problem B. We should know that finding a computable function whose runtime-bound corresponds with the complexity class and which maps instances of A to instances of B with the same yes/no-answer proofs such a reducibility.

A problem in a complexity class is said to be hard or *complete* in it iff *all* problems in this class are reducible to it. In the case of NP the corresponding bounds are polynomials in the length of the input. Reducibility is transitive on finite sets and therefore we can proof NP-completeness by using a hard reference problem: The standard MW game on a rectangular board can be modeled by special cases of MW on Graphs. Therefore the latter is not simpler and Kaye's proof of MW's completeness in NP ([2]) provides as a corollary that MWG is *NP-complete*.

Now we want to find a lower limit of NP-completeness within MWG: What is sufficient/necessary for a graph to provide hard instances of MW? For that purpose we look at other reductions to MWG. It turned out to be fruitful to regard 3-Dimensional Matching (3DM)—another NP-complete problem.

Problem 2 (3DM) *Given three finite sets of same size $k := |X| = |Y| = |Z|$ and a 3-ary relation $R \subseteq X \times Y \times Z$ on them, is there a subset $U \subseteq R$ of size $|U| = k$ which contains every element of each set X, Y and Z?*

$$\forall x \in X, y \in Y, z \in Z : \exists u_x, u_y, u_z \in U : x = \pi_1(u_x) \wedge y = \pi_2(u_y) \wedge z = \pi_3(u_z)$$

This problem can be interpreted as a party with girls X, boys Y and rooms Z and sympathies R. Could everybody be satisfied at the same time? It is easy to show that one need not demand the subset's size to be $|U| = k$

[2]A definition can be found in nearly any introduction to complexity/language theory.

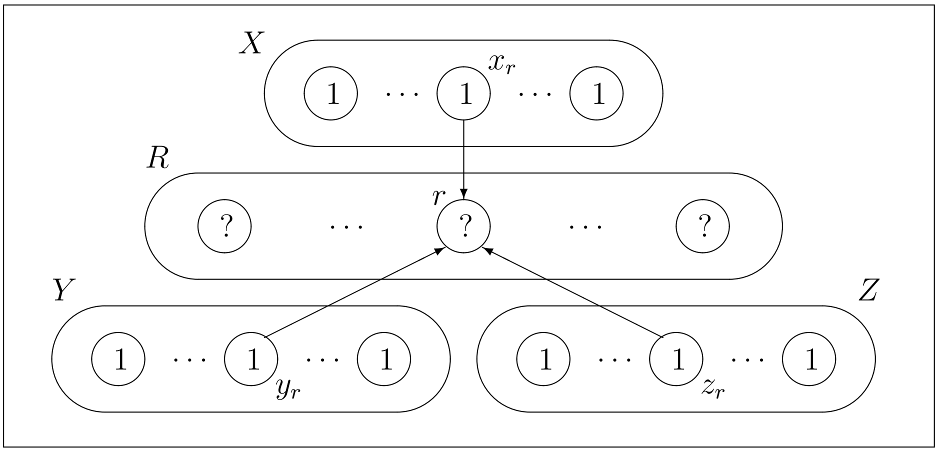

Figure 1: 3DM≤MWG Reduction

when every element of X, Y and Z must appear in (the right position of) *exactly one* tuple $u \in U$.

We are now going to show NP-completeness for special classes of MW graphs. One way to find reductions between NP-complete problems is to identify witnesses. In order to show 3DM≤MWG we use the characteristic function $\chi_{U \subseteq R}$ and build a MW graph which contains it as mines indicator μ of some unvisited fields $r \in \mathcal{F} \setminus \mathcal{D}$.

Figure 1 displays this concept: circles indicate fields labeled with numbers (1)—iff visited—and question marks (?), arrows stand for a neighborhood as known from graph drawings and the interpretation is given beside the fields. Here we can show straight forward that it's not necessary to force exactly k of the fields in question to have a mine:

$$\begin{aligned}\sum_{r \in R} \mu(r) &= \sum_{r \in R} \left(\sum_{x \in X} \mathcal{N}(x, r) \right) \mu(r) = \sum_{x \in X} \sum_{r \in R} \mathcal{N}(x, r) \mu(r) = \\ &= \sum_{x \in X} \sum_{y \in \mathcal{F}} \mathcal{N}(x, y) \mu(y) = \sum_{x \in X} \sigma(x) = \sum_{x \in X} \kappa(x) = |X| = k\end{aligned}$$

A 3-dimensional matching obviously has a consistent mine placement on the corresponding configuration and vice versa.

But it's worth to take a closer look at the configurations derived from instances of 3DM: Displayed values are restricted to 1 and we do not need the generality of the graph definition. Results shown in [3] include that the following items of a MW graph are useless—i.e. they do not decide consistency:

- edges with a *secure target* or a *source which has a mine*

- *loops*
- fields who are known to have a *mine* (if predecessors in $\mathcal{D}$ respect it)
- *isolated* fields (if $x \in \mathcal{D} \Rightarrow \kappa(x) = 0$)

And we see that the graphs we get are *directed*, *simple* (no loops or edges with the same incident vertices) and *locally star-like* (all vertices are either the source or the target of all their edges). The latter is a strong version of being bipartite for directed graphs. We also note the following bounds:

- $|\mathcal{F}| = 3k + |R| \leq 3k + k^3$
- $|\{\langle x, y\rangle \in \mathcal{F}^2 : \mathcal{N}(x, y) > 0\}| = 3|R| \leq 3k^3$
- $\mathcal{N} \leq 1$
- $\kappa \equiv 1$ and $\sigma|_{\mathcal{F}\setminus\mathcal{D}} \equiv 0$

Vertex degrees are further quantitative properties which can be subject to bounds. We easily verify $d^- \equiv 3$ for the *in-degrees*, but there is no a priori bound better than $d^+ \leq |X \times Y| \leq k^2$ for *out-degrees*. Once we have found one, we will immediately be able to state a bound for the undirected shadow-graph—which is an equivalent MWG instance—because $d^+ + d^- = \max\{d^+, d^-\}$ due to the locally star-like property.

Next we proof that $d^+ \leq 3$ can hold if more vertices are added. The structure of our graph which decides consistency is quite simple because every field 1 has some neighbors and this yields the only detail that must hold. We show that this 1-out-of-n-condition can be handled under $d^+ \leq 3$ by induction by n. There is no problem for $n = 1, 2, 3$.

Assume it is shown for n, we use the complexity reduction figured on page 6 to offer a field ${}_1x_2$ which has a mine—in a consistent mine placement—iff the field x_1 *or* x_2 has a mine and to prevent those fields from being insecure synchronously. The assumption is applied to force exactly one out of the n fields ${}_1x_2, x_3, \ldots, x_{n+1}$ to have a mine.

Two remarks should be made here: The concept of the configuration shown in figure 2 is best checked when listing *all* consistent mine placements.

x_1	x_2	?	${}_1x_2$
0	0	1	0
0	1	0	1
1	0	0	1

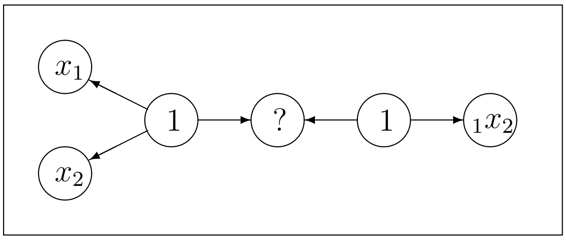

Figure 2: 1-out-of-n Complexity Reduction

And it is irrelevant whether we construct a balanced complexity reduction tree for the number of vertices added. We could bound its depth by logarithmic costs, but an *additional* amount of $4\,(n-3)$ fields and the same number of edges is used anyway.

Figure 3 shows another concept presented in [3]. This configuration can be used to cross information provided by witnesses for unvisited fields in the plane. Results from planar circuit problems provide the trick using three "equality gates". They are simulated with the pairs of 2-fields in every rectangle which force exactly 1 or 3 mines on the adjacent fields x, y and u. This makes any of them indicate whether the others are synchronously (in)secure.

After rearranging the fields of a 3DM≤MWG instance using "wires" ($x \leftarrow \bar{x} \rightarrow x \leftarrow \ldots$) and crossings, it is easy to verify that MWG is hard on *planar*, *simple* and *locally star-like* graphs with $\mathcal{N} \leq 1$, $\kappa \leq 2$, $d^- \leq 3$ and $d^+ \leq 4$.

3 SAT again

Satisfiability of Boolean formulae has been the first problem shown to be NP-complete. For *variables* x_i, $i \in \mathbf{N}$ we define *literals* x_i and $\bar{x}_i$. A *k-clause* is the disjunction $(y_0 \vee \ldots \vee y_{k-1})$ of k literals $y_i, i \in k$. A *k-formula* is the conjunction $C_0 \wedge \ldots \wedge C_{m-1}$ of finitely many k_i-clauses where $i \in m$ and $k_i \leq k$. A formula in *conjunctive normal form* (CNF) is a k-formula for some $k \in \mathbf{N}$. As usual a function $\varphi : \mathbf{N} \rightarrow \{0, 1\}$ is called *assignment* (to the variables) and extended to literals, clauses and formulae to model *Boolean logic*.

Problem 3 (SAT, kSAT) *Given a formula, is there an assignment such that it evaluates to "true"? (kSAT is restricted to k-formulae.)*

We know that SAT is NP-complete as well as kSAT for $k \geq 3$, but 2SAT is polynomially solvable. It has also been shown that the expression $\mathbf{1}(y_1, \ldots, y_n)$ ("exactly one out of these literals is true") can be modeled in

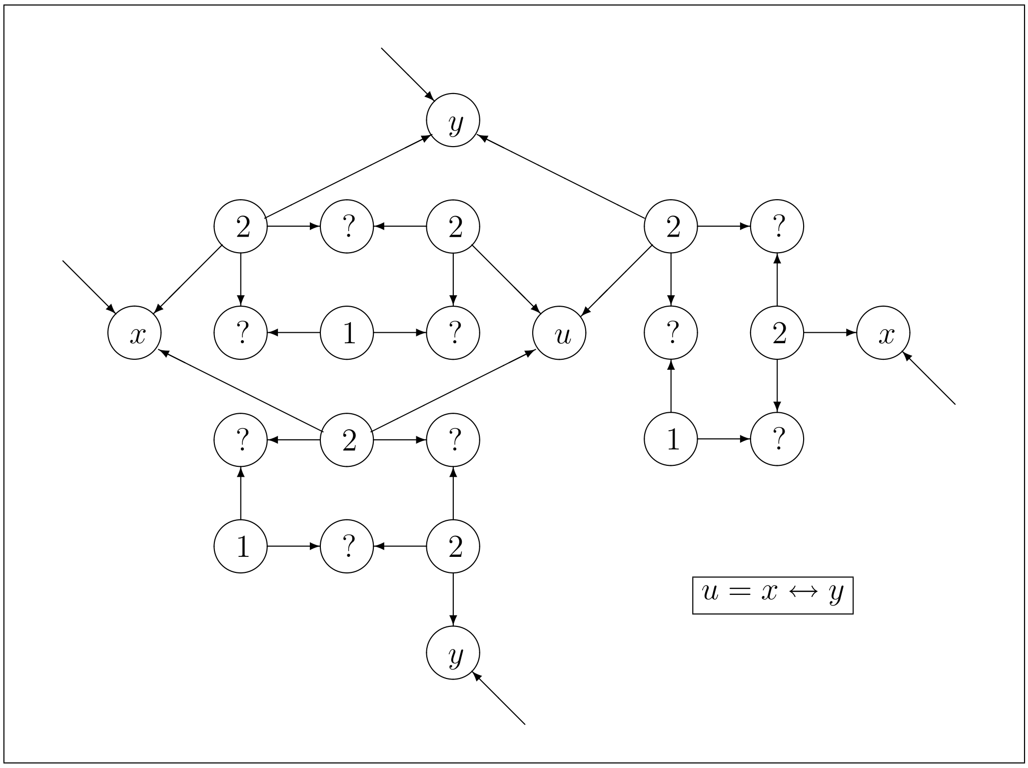

Figure 3: Plane MWG Crossing

nSAT using the literals n^2 times:

$$\mathbf{1}(y_1, \ldots, y_n) = (y_1 \vee \ldots \vee y_n) \wedge \bigwedge_{1 \le i < j \le n} (\bar{y}_i \vee \bar{y}_j)$$

Of course the expression "exactly one out of those is false" can be handled in an analogue way.

Now it is easy to reduce the MWG instances which we have constructed to SAT: Assume a directed, simple, locally star-like MW graph with $\kappa \equiv 1$, we introduce one variable per unvisited field and use $\mathbf{1}(\ldots)$ expressions to be able to identify corresponding witnesses. This reduction even yields kSAT instances with $k = \max d^+$ and every variable appears only as a positive literal and at most $\max d^-$ times in these expressions—thus at most $\max d^-$ times as a positive literal in $(\max d^+)$-clauses and $(\max d^-)\,(\max d^+ - 1)$ times negatively in 2-clauses.

This provides another approach: We do not need $\kappa \equiv 1$. Assume such a MW graph with $d^+ \le 3$, we can imagine all expressions "exactly d-out-of(-up-to)-3" for $d = 0, \ldots, 3$ because "2 out of 3" means that exactly one is "false". We find a 3-formula which is equivalent to the graph given. And of course these arguments offer a simple proof that MW graphs with $d^+ \leq 2$ are polynomially solvable via $\mathrm{MWG}_{d^+\leq 2} \leq$2SAT.

At last the locally star-like property is too strong. We can even allow undirected and mixed graphs when demanding that they are *bipartite* in such a way that the *visited fields are not separated* by the partition. We show another lemma which has already been used in analysis of standard MW games:

Lemma 4 *Given a configuration $\langle \mathcal{F}, \mathcal{N}, \mathcal{D}, \kappa \rangle$ and a consistent mine placement $\mathcal{M}$ then $\mathcal{M}_x := \mathcal{M} \bigtriangleup \{x\}$ is consistent for all $x \in \mathcal{F} \setminus \mathcal{N}(\mathcal{D})^3$.*

Proof We show that $\epsilon_x|_{\mathcal{D}} \equiv \epsilon|_{\mathcal{D}}$ holds. Note that the mine indicator for $\mathcal{M}_x$ only changes on x. Therefore $\epsilon_x(y) = \infty = \epsilon(y)$ is obvious for $y \in \mathcal{D} \cap \mathcal{M}$. On the other side $y \in \mathcal{D} \setminus \mathcal{M}$ yields

$$\epsilon(y) = \sum_{z \in \mathcal{F}} \mathcal{N}(y,z)\,\mu(z) = \sum_{z \in \mathcal{N}(\mathcal{D})} \mathcal{N}(y,z)\,\mu(z) = \sum_{z \in \mathcal{N}(\mathcal{D})} \mathcal{N}(y,z)\,\mu_x(z) = \epsilon_x(y)$$

We have shown $\epsilon_x(y) = \epsilon(y) = \kappa(y)$ for all $y \in \mathcal{D}$ and thus consistency. □

This lemma provides two corollaries immediately: On the one hand there are as many consistent mine placements with $x \in \mathcal{F} \setminus \mathcal{N}(\mathcal{D})$ being secure as those with x having a mine. This means that the probability for x having a mine—under the obvious model—is 1/2. And on the other hand we see that these $x \in \mathcal{F} \setminus \mathcal{N}(\mathcal{D})$ are unnecessary.

Now we remove all unnecessary items of this bipartite MW graph including the unvisited fields in the set of the known ones. Then we resume as above because the remaining graph is locally star-like and equivalent to the given one.

4 Summary and Outlook

For configurations on Minesweeper Graphs we found out that the graphs can be minimized to the visited fields—the domain of the partial function κ—and their neighbors without changing consistency. Above that, the neighborhood function $\mathcal{N}(x,y)$ can be set to 0 whenever $y \in \mathcal{D}$ or $x \in \mathcal{F} \setminus \mathcal{D}$, i.e. we can remove all edges with unknown source or opened target.

We found an easy way to reduce MW Graphs to SAT. Herein the proofs cover the following cases for polynomial reductions—where some of the conditions are automatically met by the equivalent minimized graphs:

[3] $\mathcal{N}(X)$ with $X \subseteq \mathcal{F}$ is the set of neighbors of X, i.e. $\{y \in \mathcal{F} : \exists x \in X : \mathcal{N}(x,y) > 0\}$

- simple, locally star-like, $\kappa \equiv 1$
- simple, locally star-like, $d^+ \leq 3$ ($d^+ \leq K \in \mathbf{N}$ will hold too)
- $\mathcal{N} \leq 1$, bipartite not separating $\mathcal{D}$; $\kappa \equiv 1$ *or* $d^+ \leq K \in \mathbf{N}$

The mentioned graphs even yield (d^+)SAT instances. Detailed regards of polynomial reductions of graphs with bounded or arbitrary κ, d^+ and $\mathcal{N}$ are going to be done. SAT solving algorithms will then be adapted for MWG.

Another intension of this article was to explore the border between polynomially solvable and NP-complete MW Graphs. Those with out-degrees bounded by $d^+ \leq 2$ reduce to 2SAT and thus are in P. More classes of simple instances (e.g. forests) are going to be found.

On the other hand we do not lose NP-completeness when requiring one of the following properties for MW Graphs:

$d^+ \leq 3$, $d^- \leq 3$, $\mathcal{N} \leq 1$, $\kappa \equiv 1$,
planar, simple, directed, locally star-like

We can even combine most of these conditions (or at least slightly weaker ones) while constructing hard instances of MWG.

Although the topic is the current subject of the author's research, this article already points out some remarkable results and ideas coming with Minesweeper on Graphs—even beyond [3]—but still seems to be far from covering easily upcoming results.

References

[1] Stephen Cook (2000): *The P versus NP Problem*, University of Toronto; Official Problem Description, Clay Mathematics Institute
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[2] Richard Kaye (2000): Minesweeper is NP-complete, *The Mathematical Intelligencer*, Volume 22, Number 2/2000, Pages 9–15
`http://web.mat.bham.ac.uk/R.W.Kaye/publ.htm`

[3] Martin Heinrich (2006): *Komplexität und Varianten von Minesweeper*, Diploma Thesis: Vienna University of Technology
`http://www.heinrichmartin.com/mw`

More references on Minesweeper—including strategy analysis and playing heuristics—are listed in the references section of [3] on pages 90f.