"Minesweeper" and spectrum of discrete Laplacians

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Abstract

The paper is devoted to a problem inspired by the "Minesweeper" computer game. It is shown that certain configurations of open cells guarantee the existence and the uniqueness of solution. Mathematically the problem is reduced to some spectral properties of discrete differential operators. It is shown how the uniqueness can be used to create a new game which preserves the spirit of "Minesweeper" but does not require a computer.

1 Paper Minesweeper: history

There is a certain class of mathematical problems which, being quite difficult to solve for an adult mathematician in their most general setting, can be understood and even be approached to in some particular cases by little kids. This paper is devoted to a problem of such a kind. Everybody knows the "Minesweeper" computer game. A subset of a rectangular table is filled with "mines" and in every spare cell the number of neighbouring mines is indicated. The general problem can be formulated as follows: given a subset of the spare part of the table with the correspondent numbers of neighbouring mines, is there a unique way to reconstruct the original distribution of mines?

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This problem in some simple cases can be used very fruitfully when teaching mathematics in primary school, since it allows to do it while playing and does not actually require a computer. The first experience of this kind belongs to the second author, who proposed for his pupils to reconstruct the distribution of mines in tables of the following type:

	2	
2	2	1

	2	
		1
2		

1	1	1

The result was very successful, our colleagues in several Aveiro schools started using such tasks.

The only practical problem we had was creating new tables so that the distribution of mines would be determined uniquely by the open area (this simplifies checking whether the solution is correct) and that the solution would not be too easy. Eventually, we found a form of the set to be open that guarantees the existence of a unique solution, finding which in most cases requires some thought. As an open set we propose to take the set of staggered cells of the initial table, in the form of a chess table. In what follows we state and prove the corresponding theorem.

2 Formal description of the game.

In the first version of the present paper [1] we restricted ourselves to the case of rectangular fields as it is in the classical computer "Minesweeper" game. Now we decide first to give a formal description of "Paper Minesweeper". The reason for such a formalization is that, as our experience with different types of fields shows, the spirit of the game is not strictly connected with the rectangularity of the field. Particularly, our experience with tables based on the triangle tiling of the plane shows that the paper version of this game encounters situations typical for the computer "Minesweeper" game.

2.1 General case

Let G be a finite undirected graph[1], with vertex set $V(G)$ and edge set $E(G)$. We say that two vertices $u, v \in V(G)$ are *neighbors*, if they are connected by an edge, that is, if $uv \in E(G)$. Thus, for every $v \in V(G)$ we can define its *neighborhood*, $N_G(v)$, as the the set of neighbors

$$N_G(v) = \{w \in V : vw \in E(G)\}.$$

A pair (A, f), where $A \subset V(G)$ and $f : A \to \{0\} \cup \mathbb{N}$, is called an *opening* if there is a subset $M \subset V(G) \setminus A$, such that for every $v \in A$,

$$f(v) = |N_G(v) \cap M|, \quad v \in A.$$

By *solving* an opening (A, f) we shall mean finding the corresponding set M. Respectively, A will be called the *set of open cells* and M will be called the *set of mines*.

Definition 1. An opening (A, f) is called a *table for "Paper Minesweeper"* if and only if it admits a unique solution.

Uniqueness of the solution gives the possibility to compare the obtained answer with the right one. Besides that, it makes the game deterministic, i.e. the presence or the absence of a mine in each cell is predetermined.

To prepare a table one can use the following algorithm: For each two subsets $A \subset V(G)$, $M \subseteq V(G) \setminus A$ we can define a function

$$f_{(A,M)}(v) = |N_G(v) \cap M|, \quad v \in A. \tag{1}$$

By construction, the set M solves the opening $(A, f_{(A,M)})$, so it remains in this case to find out when the solution is unique.

2.2 Classical computer "Minesweeper"

In the computer "Minesweeper" the table is an $m \times n$ rectangular subset R of $\mathbb{Z}^2$, $m \leq n$:

$$R = \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}.$$

[1] For the game it is supposed that V has a certain graphical representation, such that each vertex represents a cell in which either a "mine" or a number can potentially be located.

For each $\boldsymbol{i} \in R$ the set of its neighbours is defined as

$$V_{\boldsymbol{i}} = \{\boldsymbol{i} + \boldsymbol{v} \mid \boldsymbol{v} \in V\} \cap R,$$

where

$$V = \{(-1,1),(0,1),(1,1),(1,0),(1,-1),(0,-1),(-1,-1),(-1,0)\}.$$

2.3 Non-formal description

To play the game we describe in its general setting you need a playing field (for instance, printed on paper), a writing device (for instance, a pen) and a solution table. The playing field consists of "open" cells with numbers in them and "closed" cells which are to be filled by a player either with symbols of mines (crosses, for instance), or by symbols of absence of mines (dashes, for instance). The aim of the game is to fill ALL the "closed" cells in such a way that each number is equal to the number of mines in the neighbouring cells. Each playing field should be supplemented by the definition describing which cells are to be called "neighbours". According to Definition 1, a table with some of the cells filled by numbers is called *a playing field for "Paper Minesweeper"* if the distribution of mines can be restored uniquely. This particularly means that a player can check whether the obtained solution is correct by comparing it with the solution table.

3 Statement of the main theorem

The first result in this direction appeared within the framework of a project of the second author's department after proposing school students of the 8-th form to reconstruct the distribution of mines in a table $2 \times n$ with the upper string as A.[2] At that time the following theorem was proved:

Theorem 1. *Suppose that*

$$R = \{(i,j) \mid 1 \le i \le 2, 1 \le j \le n\}$$

and that $n+1$ is not divisible by 3. let A be one of the two strings of R:

$$A = \{(1,i) \in R \mid i = 1, \ldots, n\}.$$

[2]Special thanks to Prof. Ana Breda for organization of this project

Then for every $M \subset R \backslash A$ *the opening*

$$(A, f_{A,M})$$

admits only one solution.

To prove Theorem 1 we used inductive calculation of 3-diagonal determinants. The argument is quite simple, so we skip it. Later on we obtained a more interesting theorem, which makes the main result of this paper:

Theorem 2. *Suppose that*

$$R = \{(i,j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}, \quad m \leq n,$$

the numbers $n+1$ *and* $m+1$ *are coprime. Let* A *be the subset of* R *in the form of a chess table:*

$$A = \{(i,j) \in R \mid i+j \text{ is even}\}.$$

Then for every $M \subset R \backslash A$ *the opening*

$$(A, f_{A,M})$$

admits only one solution.

3.1 Tables based on the triangle tiling of the plane.

Let the neighbours of $\boldsymbol{i} = (i,j) \in R$ be defined by the following rule:

$$V_{\boldsymbol{i}} = \{\boldsymbol{i} + \boldsymbol{v} \mid \boldsymbol{v} \in V_t^{\boldsymbol{i}}\} \cap R,$$

where

$$\begin{aligned} V_t^{\boldsymbol{i}} = \{&(0,1),(1,1),(1,0),(-1,0),(1,-1),(0,-1),(-1,-1),\\ &(-1,1),(0,2),(0,-2),((-1)^{i+j},2),((-1)^{i+j},-2),\}, \quad \boldsymbol{i} = (i,j). \end{aligned} \tag{2}$$

If we associate naturally the vertices of this graph with triangles in the triangle tiling of the plane, then this rule means that two triangles should be called neighbours if they have at least one common vertex (see fig. 7).

Theorem 3. *Let $R = \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$, and let $(m + 1)$ and $(n+1)$ be not divisible by 4. Let neighbours be defined by the set (2). Suppose that A is a subset of R in the form of a chess table:*

$$A = \{(i, j) \in R \mid i + j \text{ even}\}.$$

Then for each set of mines $M \subset R \backslash A$ the opening $(A, f_{A,M})$ admits only one solution.

It should be noticed that, due to the larger number of neighbours, games based on such tables have a higher level of complexity. It is curious that in these tables a player regularly encounters situations similar to those in the computer game.

4 An algorithm of table making.

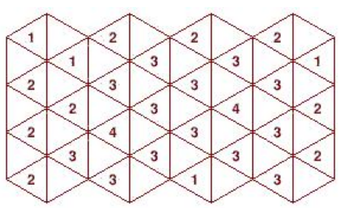

Figure 1: Table on base of equilateral triangles.

Before proving Theorem 2 we apply it to describe an algorithm of generating openings with unique solutions. In most cases the resulting opening will be nontrivial to solve.

Suppose that m and n satisfying the conditions of Theorem 2 are chosen. Set M to be empty initially.

1. For every $\boldsymbol{i} \in R \backslash A$ we execute a Bernoulli test, and if the result is 1, we add $\boldsymbol{i}$ to M. The probabilities p and q in the test can be taken equal to $1/2$.

2. After having run through all the elements of $R \backslash A$ we define $f_{A,M}$ by (1) and fill all the cells $\boldsymbol{i} \in A$ with the values $f_{A,M}(\boldsymbol{i})$.

An opening with a unique solution is ready. Enjoy the game! Here are some examples of openings that can be obtained this way:

1		2
	2	
1		1
	1	

	1		1		1
2		2		1	
	3		2		1
1		3		2	
	1		2		1

	1		1		1
1		1		1	
	1		1		1
1		1		2	
	1		1		1

(3)

The solutions can be found at the end of the paper.

The first step of the algorithm can be modified in order to avoid the situation when for some $\boldsymbol{i} \in A$ one has

$$\text{either} \quad f_{(A,M)}(\boldsymbol{i}) = \sharp(V(\boldsymbol{i})), \quad \text{or} \quad f_{(A,M)}(\boldsymbol{i}) = 0. \tag{4}$$

We propose to fill up the first row independently and then fill all the other rows, starting with the second one and on downwards, filling each row from left to right and taking into account the distribution of mines already placed.

Consider the following pseudocode:

for i:=2 to m
for j:=1 to n
if $f_{(A,M)}(i-1,j) == 0$ then put a mine into the cell (i,j)
if $f_{(A,M)}(i-1,j) == (\sharp(V(i-1,j)) - 1)$ then leave the cell (i,j) empty
end
end

A table obtained by the application of this algorithm can have cells satisfying the condition (4) only in the last row. Thus, in general, such tables are already more complicated. But if we want the inequality $0 < f_{(A,M)}(\boldsymbol{i}) < \sharp(V(\boldsymbol{i}))$ to be valid for every $\boldsymbol{i} \in A$ we should apply the previous algorithm for all the rows but the last one. And for the last row we should apply the following algorithm:

for j:=2 to n
if $f_{(A,M)}(m-1,j) == 0$ then put a mine into the cell (m,j)
if $f_{(A,M)}(m-1,j) == (\sharp(V(m-1,j)) - 1)$ then leave the cell (m,j) empty
if $f_{(A,M)}(m,j-1) == 0$ then put a mine into the cell (m,j)
if $f_{(A,M)}(m,j-1) == (\sharp(V(m,j-1)) - 1)$ then leave the cell (m,j) empty
end

Note, however, that this procedure guarantees that there is no cell satisfying (4) only in the case when the number of columns plus the number of rows is even. In the other case (4) can hold for the cell (m,n).

These algorithms were realized in
http://www2.mat.ua.pt/jpedro/minesweeper/the-tablep.htm

5 Proof of Theorem 2.

Consider the opening (A, f). Denote by $\mathcal{X}$ the set of characteristic functions $\{0,1\}^{R\setminus A}$. We remind that there is a natural bijection between $\mathcal{X}$ and the set of all the subsets of $R\backslash A$. Namely, each $M \subset R\backslash A$ corresponds to the function $x_M \in \mathcal{X}$ defined by the equality

$$x_M(\boldsymbol{i}) = \begin{cases} 1, & \boldsymbol{i} \in M, \\ 0, & \boldsymbol{i} \notin M, \end{cases} \qquad \boldsymbol{i} \in R\backslash A. \tag{5}$$

Due to this bijection we can say that an element of $\mathcal{X}$ solves the opening meaning that its support does. Now, the condition that a function $x \in \mathcal{X}$ solves the opening (A, f) can be written as a system of linear equations:

$$\sum_{\boldsymbol{j} \in V_{\boldsymbol{i}} \cap (R\backslash A)} x(\boldsymbol{j}) = f(\boldsymbol{i}), \quad \boldsymbol{i} \in A. \tag{6}$$

Since

$$A = \{(i, j) \in R : i + j \text{ is even}\}$$

we can rewrite (6) as

$$\sum_{\boldsymbol{j} \in \{(i-1,j),(i+1,j),(i,j-1),(i,j+1)\} \cap R} x(\boldsymbol{j}) = f(\boldsymbol{i}), \quad \boldsymbol{i} = (i, j) \in A. \tag{7}$$

We fix arbitrary orders on the sets $A, R\backslash A$. Given these orders we can consider the matrix of the system 7 and denote it by E. This matrix is square, since mn is even and therefore $|A| = |R\backslash A|$. To prove the uniqueness of the solution of (7) it suffices to show that E is invertible. Since the invertibility of E does not depend on the order on R, we shall not specify the latter.

Denote

$$P = \left\{ (x, y) = \left(\frac{\pi k}{m+1}, \frac{\pi l}{n+1} \right), \quad (k, l) \in R \right\} \subset S^1 \times S^1.$$

Consider the real space $\mathcal{L} = L_2(P)$. It is well known that the system $\{\sin kx \sin ly, (k, l) \in R\}$ forms a basis in $\mathcal{L}$. Consider the operator L acting on $L_2(P)$ as multiplication by $2(\cos x + \cos y)$:

$$\frac{1}{2}(Lg)(x, y) = \Big(\cos x + \cos y \Big) g(x, y), \quad g \in \mathcal{L}, \quad (x, y) \in P.$$

Note that

$$L(\sin ix \sin jy) = \sum_{(k,l) \in \{(i-1,j),(i+1,j),(i,j-1),(i,j+1)\} \cap R} \sin kx \sin ly, \quad (i, j) \in R.$$

Denote

$$\widehat{\mathcal{L}}_1 = \mathrm{span}\{\sin kx \sin ly, (k,l) \in R, k+l \text{ is odd}\} \subset \mathcal{L},$$
$$\widehat{\mathcal{L}}_2 = \mathrm{span}\{\sin kx \sin ly, (k,l) \in R, k+l \text{ is even}\} \subset \mathcal{L}.$$

It is easy to see that

$$L(\widehat{\mathcal{L}}_2) \subset \widehat{\mathcal{L}}_1, \quad L(\widehat{\mathcal{L}}_1) \subset \widehat{\mathcal{L}}_2.$$

Denote

$$L_{12} = L|_{\widehat{\mathcal{L}}_2}, \quad L_{21} = L|_{\widehat{\mathcal{L}}_1}.$$

The matrix of the operator L_{21} coincides exactly with the transpose of E (it is supposed that bases in $\widehat{\mathcal{L}}_1$ and $\widehat{\mathcal{L}}_2$ are chosen in accordance with the orders on A and $R\backslash A$, which were used to define the matrix E). Thus it suffices to show that L_{21} is invertible.

The subspace $\widehat{\mathcal{L}}_1$ is invariant under the action of L^2. The restriction of L^2 to $\widehat{\mathcal{L}}_1$ coincides with $L_{12}L_{21}$. Thus, if we prove that L^2 is invertible, so will be both L_{21} and L_{12}.

The spectrum of L coincides with the set

$$\Big\{2\Big(\cos x + \cos y\Big) \;\Big|\; (x,y) \in P\Big\}.$$

To show this it suffices to notice that the functions defined by the equalities

$$\chi_{x,y}(u,v) = \delta_{xu}\delta_{yv}, \quad (x,y) \in P,$$

are eigenfunctions of L:

$$\frac{1}{2}L\chi_{x,y} = \Big(\cos x + \cos y\Big)\chi_{x,y}.$$

We get that 0 is an eigenvalue of L if and only if there are integers k, l, such that

$$\frac{\pi k}{m+1} - \frac{\pi l}{n+1} = \pm\pi, \quad 1 \le k \le m, \quad 1 \le l \le n.$$

But this is not so, since $m+1$ and $n+1$ are coprime.

Thus, L is invertible, which proves the theorem.

6 Proof of Theorem 3

The proof is quite similar in this case, save that L now acts as

$$L(\sin ix \sin jy) = \sum \sin(i+k)x \sin(l+j)y, \quad (i,j) \in R, \tag{8}$$

where the summation is taken over all the pairs

$$(k,l) \in \{(0,1),(1,0),(-1,0),(0,-1),((-1)^{i+j},2),((-1)^{i+j},-2)\} \cap R.$$

This operator is not symmetrical and thus cannot be represented as multiplication by a function in $L_2(P)$. Same as we did before, we can define spaces $\widehat{\mathcal{L}}_1$ and $\widehat{\mathcal{L}}_2$, and operators L_{12} and L_{21}. Now our objective is also to find out whether L_{21} has maximal rank. Indeed, by our definition of sets A and (2) we have $|R\backslash A| = |A| - 1$ in case of odd mn and $|R\backslash A| = |A|$ in case of even mn (see pict). In the latter case maximality of rank means invertibility of L_{21}.

By straightforward calculations we obtain that LL^* acts as

$$LL^*(\sin ix \sin jy) = \sum_{\{(k,l)\in V_{LL^*}\cap R} \sin(i+k)x \sin(l+j)y, \quad (i,j) \in R, \tag{9}$$

where

$$\begin{aligned}\left(\frac{1}{2}LL^*f\right)(x,y) = {}& [3 + 3\cos(2x) + \cos(4x) + \cos(2y) + \\ & + 6\cos(x)\cos(y) + 2\cos(2x)\cos(2y) + \\ & + 2\cos(3x)\cos(y)]f(x,y), \qquad (x,y) \in P\end{aligned}$$

or

$$\frac{1}{4}(LL^*f)(x,y) = \left|\cos(x) + \cos(y) + e^{ix}\cos(2y)\right|^2 f(x,y), \quad (x,y) \in P.$$

We have

$$\begin{cases} \cos(x) + \cos(y) + \cos(x)\cos(2y) = 0 \\ \sin(x)\cos(2y) = 0 \end{cases} \tag{10}$$

We cannot have $\sin(x) = 0$, since $(x,y) \in P$, and so, $x = \pi k/m+1,\ k = 1,\dots,m$. Hence $\cos(2y) = 0$, which means that $n+1$ is a multiple of 4, since

$$2\frac{\pi l}{n+1} = \frac{\pi}{2}, \quad l = 1,\dots,n.$$

Thus, the second equation of (10) is equivalent to $\cos 2y = 0$. This, together with the first equation of (10), implies that

$$|\cos x| = \frac{\sqrt{2}}{2}, \quad (x, y) \in P.$$

Hence $m + 1$ is a multiple of 4. Theorem 3 is proved.

7 Solutions of (3).

1	*	2
-	2	*
1	-	1
*	1	-

-	1	-	1	*	1
2	*	2	-	1	-
*	3	*	2	-	1
1	-	3	*	2	*
-	1	*	2	-	1

-	1	-	1	*	1
1	*	1	-	1	-
-	1	-	1	-	1
1	-	1	*	2	*
*	1	-	1	-	1

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[1] Oleg German, Evgeny Lakshtanov, *"Minesweeper" without a computer*, arXiv:0806.3480v1 [cs.DM]

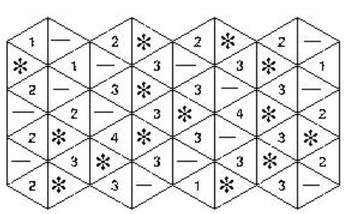

Figure 2: Solution for the table presented on Figure 1.