# The Minesweeper game: Percolation and Complexity

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**Abstract**

We study a model motivated by the minesweeper game. In this model one starts with percolation of mines on the sites of the lattice $\mathbf{Z}^d$, and then tries to find an infinite path of mine free sites. At every recovery of a free site, the player is given some information on the sites adjacent to the current site. We compare the parameter values for which there exists a strategy such that the process survives to the critical parameter of ordinary percolation. We then prove improved bounds for these values for the same process, when the player has some complexity restrictions in computing his moves. Finally, we discuss some monotonicity issues which arise naturally for this model.

## 1 Introduction

### 1.1 The finite game

The minesweeper is a popular computer game. The game is played on a finite square grid $G_N = \{1, \ldots, N\} \times \{1, \ldots, N\}$. Let $M \subset G_N$ be the set of *mines*; an element $v \in M$ is called a *mine*; an element in $v \in G_N \setminus M$ is called a *free site*. The set $M$ is unknown to the player, and the aim of the game is to find this set $M$. At every step, the player is allowed to guess if a certain site $v \in G_N$, which currently has unknown status, is a free site. If $v \in M$ (i.e. the site is a mine) then the player loses. Otherwise, the player is given the number of sites adjacent to this site which are occupied by mines: $|\{v' \in G_N : d_\infty(v, v') = 1, v' \in M\}|$, and then plays a new move.

We remark that in the original game, when the game begins, the player is also given the number of mines: $|M|$. Since we are interested in infinite variants of this game, we simplify the model and ignore this extra information.

### 1.2 The infinite game

In this paper we consider an infinite variant of this game. Throughout this paper when we refer to $\mathbf{Z}^d$ as a graph, we mean the graph $\mathbf{Z}^d$ with the $l_\infty$ neighboring structure:

$$(\mathbf{Z}^d, \{(v, u) \in \mathbf{Z}^d \times \mathbf{Z}^d \text{ such that } d_\infty(v, u) = 1\}).$$

We recall that *site percolation* with parameter $p$ on $\mathbf{Z}^d$ is the random subgraph $\omega$ of $\mathbf{Z}^d$ which satisfy $\mathbf{P}[v \in \omega] = p$, independently for all sites $v \in \mathbf{Z}^d$. For such $\omega \subset \mathbf{Z}^d$ and $v \in \mathbf{Z}^d$ we say that $v$ is *open* (this is the standard term), or *free* (this is the term which corresponds to the mine sweeper game) if $v \in \omega$; otherwise, we say that $v$ is *closed*, or

$v$ is a *mine*. We note that the set of mines, $\mathbf{Z}^d \setminus \omega$, may be obtained as the set of open vertices in $1-p$ percolation.

The set $\omega$ is unknown to the player and at every step the player is requested to find some new site $v \in \mathbf{Z}^d$ which is a free site. If $v \notin \omega$ the player loses. Otherwise, he is given the number: $|\{v' \in \mathbf{Z}^d : d_\infty(v, v') = 1 \text{ and } v' \notin \omega\}|$ and then plays a new step.

If $0 < p < 1$ and the player is requested to find the set of all mines, $\mathbf{Z}^d \setminus \omega$, then he will almost surely die. This follows from the fact that if $0 < p < 1$, then a.s. there are infinitely many sites which have only mine neighbors. A more realistic aim for the player is to play an infinite sequence of steps. This leads to the following definitions:

**Definition 1.1** *1. A* **game-configuration** *is a triplet $(\omega, \mu, I)$ where $\omega$ is the set of free sites, $\mu \subset \omega$ is the finite subset of free sites which are known to the player and $I : \mu \to \{0, \ldots, 3^d - 1\}$ is the function which satisfies*

$$I(v) = |\{v' : d_\infty(v', v) = 1 \text{ and } v' \notin \omega\}|; \tag{1}$$

*i.e. $I$ is the information known to the player.*

2. *We call $S$ a* **strategy**, *if $S$ is a function from all pairs $(\mu, I)$ as above to sites in $\mathbf{Z}^d$ which satisfies $S(\mu, I) \notin \mu$ (i.e. $S$ always picks a new site).*

3. *The* **game determined by** *$S$, denoted $\mathcal{G}(S)$, is a sequence of game configurations*

$$\mathcal{G}(S) = (c_n)_n = ((\omega, \mu_n, I_n))_n$$

*which is defined in the following way: $\omega$ is chosen according to p-percolation. We define $\mu_n$ and $I_n$ recursively. We start by defining $\mu_0 = \emptyset$ and let $I_0$ be the empty function. Now we continue recursively,*

- *If $S(\mu_n, I_n) \notin \omega$ the sequence is determined to be $(c_0, \ldots, c_n)$.*
- *If $S(\mu_n, I_n) \in \omega$ we set $\mu_{n+1} = \mu_n \cup \{S(\mu_n, I_n)\}$, let $I_{n+1}$ satisfy (1) and continue recursively.*

4. *For a strategy $S$ define*

$$\theta_S(p) = \mathbf{P}_p[|\mathcal{G}(S)| = \infty],$$

*where $\mathbf{P}_p$ is the Bernoulli measure on $\mathbf{Z}^d$ for which a site is free with probability $p$. Define:*

$$\theta(p) = \sup_S \theta_S(p).$$

*Thus, $\theta_S(p)$ is the probability of winning the game playing $S$, and $\theta(p)$ is the probability of winning the game playing an "optimal" strategy.*

5. *Finally, we define,*

$$W(d) = \{p \in [0, 1] : \theta(p) > 0\}.$$

*$W(d)$ is the set of all probabilities for which one can win the game.*

**Proposition 1.2** *Set*

$$Y = \#\{n : S_n(\mu_n, I_n) \notin \partial\mu_n\}$$

*where $\partial A$ is defined as $\{v \in \mathbf{Z}^d : d_\infty(v, A) = 1\}$. Then for all $S$ we have*

$$\mathbf{P}[Y \geq n + 1] \leq p^n.$$

**Proof:** For all $v \notin \partial\mu_n$ and all $I_n$ which satisfy (1), $\mathbf{P}[v \in \omega | I_n] = p$. □

Proposition 1.2 implies that a.s. during the game the player only plays inside a finite number of clusters (connected components of $\omega$). It is therefore natural to compare the set $W(d)$ to the set of parameters $p$ for which an infinite cluster exists.

Our first result bounds the values of $p$ for which we can win the game. We denote by $p_c(d)$ the critical value for percolation on $\mathbf{Z}^d$ (with the $l_\infty$ neighboring structure). Thus for $p > p_c(d)$, with probability 1 there exists an infinite path of free sites. For $p < p_c(d)$ such a path does not exist a.s. Using the enhancement results by Aizenman and Grimmett (see [2]) we prove:

**Theorem 1.3** *For all $d \geq 2$ there exists an $\epsilon = \epsilon(d) > 0$ such that*

$$[1-\epsilon, 1] \subset W(d) \subset [p_c(d)+\epsilon, 1].$$

Thus, when the density of mines is small, the player may get out of the mine-field. On the other hand, there is an interval of densities for which there exists an infinite path which is free of mines, yet the player is doomed to die.

Another natural question is: what are the connections between the complexity of the strategies used and the sets of winning probabilities? In the following definition we give a formulation of complexity in this context.

**Definition 1.4** *Let $(\omega, \mu, I)$ be a game configuration. We call $v \in \mathbf{Z}^d$ a* **trivially free** *site for the configuration, if $v \in \omega \setminus \mu$ and there exists $u \in \mu$ which satisfies $d_\infty(v, u) = 1$ and $I(u) = 0$.*

*We call a strategy $T$ a* **trivial strategy** *if the following holds:*

1. *For a game configuration $(\omega, \mu, I)$ which has a trivially free site, $T(\mu, I)$ is a trivially free site.*
2. *For $(\omega, \mu, I)$ for which there exists no trivially free site, we have $T(\mu, I) \notin \partial\mu$. (In other words: when there are no trivially free sites, $T$ chooses a site on which it has no information whatsoever)*

As part of Theorem 1.3, we prove:

**Proposition 1.5** *For all $d \geq 2$ there exists an $\bar{\epsilon}(d) > 0$ such that if $T$ is a trivial strategy then,*

$$(1-\bar{\epsilon}(d), 1] \subset \{p : \theta_T(p) > 0\} \subset [1-\bar{\epsilon}(d), 1].$$

This result says that even the lowest complexity algorithms suffice for winning for low mines density. Rephrasing Proposition 1.5 in terms of the original game, we get that there exists $\bar{\epsilon}(d) > 0$ such for $p \in (1-\bar{\epsilon}(d), 1]$, we may reconstruct an infinite path with positive probability on the "first click". On the other hand, we prove that by utilizing more complex strategies, we may win in higher mines densities:

**Theorem 1.6** *For all $d \geq 2$ there exists an $\epsilon(d) > \bar{\epsilon}(d)$ such that*

$$(1-\epsilon(d), 1] \subset W(d).$$

The proof of this theorem uses again the method of enhancements. However, some additional work is needed. In particular, we devise some hand-tailored combinatorial designs in order to prove some strict inequalities. We believe that there is a hierarchy of results generalizing Theorem 1.6 by saying that when one uses more information during the game, the set of winning probabilities strictly increases. See Conjecture 3.3 for the exact formulation.

Unfortunately, we do not know if the set $W(d)$ is connected. In Section 4 we discuss some examples which are related to this monotonicity problem.

It seems that the minesweeper game introduces many interesting problems in diverse fields. Some of these problems are considered in this paper. For different perspectives, see [1] and [6].

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## 2 Comparison to Percolation

**Proof of Proposition 1.5:** Let $S$ be a trivial strategy, and assume that $\theta_S(p) > 0$. Call a game configuration $c = (\omega, \mu, I)$ *critical*, if there exists no site which is trivially free for $c$. If $c$ is critical then since $S$ is trivial, it follows that $\mathbf{P}[S(\mu, I) \in \omega] = p$. Thus if $\mathcal{G}(S) = (c_n)$, then the probability that more than $t$ of the $c_n$ are critical is bounded by $p^t$. It follows that given $|\mathcal{G}(S)| = \infty$, there are a.s. only finite number of critical $c_n$. Let $A$ be the event that the percolation cluster contains an infinite connected subgraph of vertices $v$, such that all vertices at $l_\infty$ distance 1 from $v$, are also in the percolation cluster. From the argument above it follows that $\theta_S(p) > 0$ implies $\mathbf{P}_p(A) > 0$.

On the other hand, suppose that $\mathbf{P}_p(A) > 0$ and let $S$ be a trivial strategy. It is then easy to see that $\theta_S(p) > 0$.

We achieved that $\theta_S(p) > 0$ if and only if $\mathbf{P}_p(A) > 0$. In other words, writing $\bar{W}(d) = \{p : \theta_S(p) > 0\}$, we obtain

$$\{p : \mathbf{P}_p(A) > 0\} = \bar{W}(d).$$

Since $A$ is an increasing event there exists an $\epsilon_0 = \epsilon_0(d) \geq 0$ such that

$$(1 - \epsilon_0, 1] \subset \{p : \mathbf{P}_p(A) > 0\} \subset [1 - \epsilon_0, 1].$$

In order to conclude we show that $\epsilon_0 > 0$.

Divide $\mathbf{Z}^d$ into cubes of side length 3 centered at $3\mathbf{Z}^d$. Declare such a cube to be open if it contains no mines. Now, consider the percolation process on the cubes with the $l_1$ neighborhood structure. This is a Bernoulli percolation with parameter $p^{3^d}$. Therefore, there exists an $\epsilon > 0$ such that this percolation process survives for all $p \in [1 - \epsilon, 1]$. Note that the event $A$ occurs whenever there is an infinite cluster of open cubes (see Figure 1). It follows that $\epsilon_0 \geq \epsilon > 0$ as needed. □

In order to prove that $W(d) \subset [p_c(d) + \epsilon, 1]$, we use the method of strict inequalities which was developed by Aizenman and Grimmett ([2]), after Menshikov ([7]). We follow some of their notation below.

A boolean function $f : \{0,1\}^{\mathbf{Z}^d} \to \{0,1\}$ is called *local* if there exists a finite set $\tau$ such that, whenever $\omega_1 \cap \tau = \omega_2 \cap \tau$, $f(\omega_1) = f(\omega_2)$. For a local function $f$ we define $f_v(\omega)$ by

$$f_v(\omega) = \begin{cases} \{v\} & \text{if} \quad f(\omega - v) = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

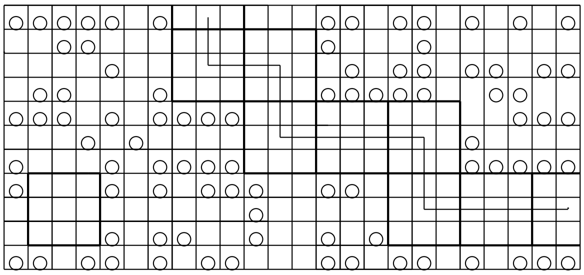

Figure 1: Sketch of Proposition 1.5

where $\omega - v$ is the translation of $\omega$ by $-v$. Define the *diminishment* procedure $D_f$ which corresponds to the local function $f$ to be the function which maps the configuration $\omega$ to the configuration $\tilde{\omega}$ defined by:

$$\tilde{\omega} = \omega \setminus \bigcup_{v \in \mathbf{Z}^d} f_v(\omega).$$

A diminishment procedure is called *essential* if there exists a configuration $\omega$ which contains a doubly-infinite path, but $\tilde{\omega}$ does not contain such a path. We need the following theorem in order to prove that $W(d) \subset [p_c(d) + \epsilon, 1]$.

**Theorem[Aizenman,Grimmett]:** *Consider the following two stage procedure. Perform site percolation with parameter $p$, and on the resulting configuration perform some essential diminishment $D$. Then, there exists an $\epsilon > 0$ such that for all $p < p_c(d) + \epsilon$, the set of resulting sites does not contain an infinite cluster a.s.*

**Lemma 2.1** *For all $d \geq 2$ there exists an $\epsilon = \epsilon(d) > 0$ such that*

$$W(d) \subset [p_c(d) + \epsilon, 1].$$

**Proof:** Look at the percolation of free sites with density $p$. Consider the diminishment procedure which sets the site $v$ closed whenever it observes the configuration $\sigma + v$ on $\prod_{i=1}^{d}[v_i - 5, v_i + 5]$ in which the free sites are exactly $\{u : u_1 = v_1, u_2 = v_2 \ldots, u_{d-1} = v_{d-1}\}$ (see Figure 2 ; $\sigma$ is the configuration on $[-5, 5]^d$ which has as free sites: $-5e_d, \ldots, 5e_d$). This is an essential diminishment. Therefore, there exists an $\epsilon > 0$ such that whenever $p < p_c(d) + \epsilon$ the diminished configuration $\tilde{\omega}$ does not contain an infinite cluster of free sites with probability 1.

We claim that $W(d) \subset [p_c(d) + \epsilon, 1]$. Suppose that $p \in W(d)$ and let $S$ be a strategy with $\theta_S(p) > 0$. Let $\mathcal{G}(S)$ be the game determined by $S$, and let $v^n$ satisfy $v^n \in \mu_{n+1} \setminus \mu_n$. By Proposition 1.2, a.s. for all but finite number of $n$'s, $v^n \in \partial \mu_n$.

Similarly, suppose that $v = v^n$ and the configuration in $\prod_{i=1}^{d}[v_i - 5, v_i + 5]$ is $\sigma + v$. Note that

$$\mathbf{P}[v^n \in \omega \text{ and } v^n + e_1 \notin \omega | I_n] = \mathbf{P}[v^n \notin \omega \text{ and } v^n + e_1 \in \omega | I_n].$$

Therefore, there exits $\delta > 0$ such that $\mathbf{P}[v^n \in \omega | I_n] \leq 1 - \delta$. It now follows that a.s. the sequence $v^n$ contains only finitely many elements $v$ such that $\sigma + v$ is the configuration of $\prod_{i=1}^{d}[v_i - 5, v_i + 5]$.

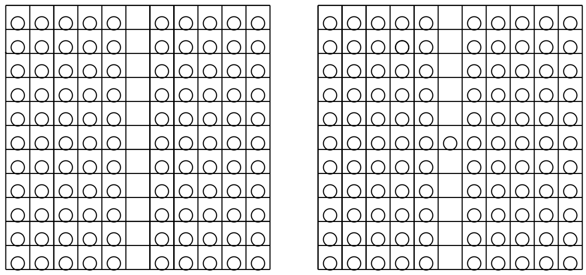

Figure 2: The diminishment procedure

Since $\theta_S(p) > 0$, we see that the diminished percolation survives with positive probability. Therefore $p \geq p_c(d) + \epsilon$ as needed. □

**Proof of Theorem 1.3**: Follows from Lemma 2.1 and Proposition 1.5. □

## 3 Simple and Complex Strategies

In this section we prove Theorem 1.6 for $d = 2$, the proof for general $d$ being similar. The first step is to translate the problem to one which does not involve strategies, but only the structure of the percolation cluster.

**Definitions 3.1:** *Call a site $v$* **wide free** *if $v$ is a free site and all sites $w$ with $d_\infty(v, w) \leq 1$, are free sites. Call a site $v$* **almost wide free**, *if it is a wide free site, or if it has a single mine neighbor $w = (w_1, w_2)$, and the configuration of mines and free sites in $[w_1 - 6, w_1 + 6] \times [w_2 - 4, w_2 + 4]$ is an exact copy of Figure 3, where the sites labeled by o are mines and sites which are not labeled, or labeled by $V$, are free sites (i.e., the sites in Figure 3 that are labeled by $V$ are not wide free but are almost wide free). Note that every wide free site, is an almost wide free site.*

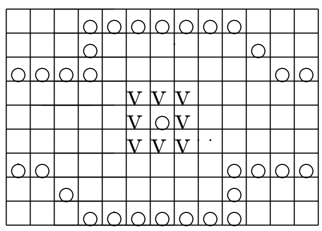


Figure 3: Definition of almost wide free sites

Given $\omega$, let $\omega_T$ be defined as follows.

- If there exists a site $v$ such that $d_\infty(0, v) < 5$ and $v$ is a mine, set $\omega_T = \emptyset$.
- Otherwise, $\omega_T$ is the connected component of wide free sites for which $0 \in \omega_T$.

We let $A_T = \{\omega : |\omega_T| = \infty\}$.

Given $\omega$, let $\omega_A$ be defined in the following way:

1. If there exists a site $v$ such that $d_\infty(0,v) < 5$ and $v$ is a mine, set $\omega_A = \emptyset$.

2. Otherwise, $\omega_A$ is the connected component of almost wide free sites which contains 0.

We let $A_A = \{\omega : |\omega_A| = \infty\}$.

The proof of Proposition 1.5 implies that if $S$ is a trivial strategy, then the set of probabilities for which one can win the game using $S$, $\bar{W}(d)$, does not depend on $S$, and satisfies $\bar{W}(d) = \{p : \mathbf{P}_p(A_T) > 0\}$. In particular, $\bar{W}(d)$ is an interval. We will show that $W(d)$ contains an interval which is longer than $\bar{W}(d)$, by showing that $\{p : \mathbf{P}_p[A_A] > 0\}$ contains such an interval, and the following lemma.

**Lemma 3.1**

$$\{p : \mathbf{P}_p[A_A] > 0\} \subset W(d). \tag{2}$$

**Proof:**. Let $S$ be the following strategy.

1. At the first step, try to recover 0.

2. If there exists a trivially free site, then recover it.

3. When a site $v$ is recovered together with all the neighbors of $v$ but one, and it is given that $v$ has a single mine neighbor, then "mark" this neighbor as a mine.

4. If $v$ has a single mine neighbor, and a neighbor $u$ of $v$ is marked as a mine, recover all other neighbors of $v$.

We claim that when $A_A$ holds, $\mathcal{G}(S)$ is infinite. This implies (2). Assume that $A_A$ holds. Therefore, there exists an infinite self avoiding path of almost wide frees sites: $\varphi = (0 = v^0, v^1, v^2, \ldots)$. We will show by induction that $S$ can recover all these sites (and some more).

Since 0 is a free site, $0 = v^0$ is recovered. Suppose that $S$ recovered $v^0, \ldots, v^n$. If $v^n$ is a wide free site, then $v^{n+1}$ as a neighbor of $v^n$, is a trivially free site. Therefore $S$ recovers $v^{n+1}$.

A more interesting case is when $v^n$ is a site as one of the $V$'s in Figure 3. Let $(x_0, y_0)$ be the location of the mine which is adjacent to $v^n$.

Since $\varphi$ is connected, and by the assumption that all $v$ with $d(v,0) < 5$, are free sites, it follows that there exists $k < n$, such that $v^k = (x_0 - 5, y_0)$, or there exists $k < n$ such that $v^k = (x_0 + 5, y_0)$. Assume without loss of generality that $v^k = (x_0 - 5, y_0)$.

Given $v^k$, $S$ can easily recover all the sites in $[x_0 - 5, x_0 - 1] \times [y_0 - 1, y_0 + 1]$ (as trivially free sites). Now it may recover $(x_0 - 2, y_0 - 2)$ as a trivially free site, and then similarly all the remaining sites in $[x_0 - 2, x_0 + 2] \times [y_0 - 3, y_0 - 1]$.

Note that now the site $(x_0 - 1, y_0 - 1)$ and all its neighbors but one are recovered. Moreover, $(x_0 - 1, y_0 - 1)$ has a single mine neighbor. We therefore mark (using rule 3 of $S$) $(x_0, y_0)$ as a mine. After marking $(x_0, y_0)$ as a mine, it is easy to recover (using rule 4) all the neighbors of $(x_0, y_0)$ and their neighbors. One of these sites must be $v^{n+1}$. $\square$

The next step of the proof is to introduce a new parameter $s$. Consider Bernoulli percolation with parameter $s$ which is independent of the $p$ percolation of free sites. We now consider sites which are either

- wide-free, or

- almost-wide-free with single mine neighbor which is open in the $s$-percolation.

We call such sites *good* sites. Given $\omega$, let $\omega'$ be defined as follows:

1. If there exists a site $v$ such that $d(0,v) < 5$ and $v$ is a mine, set $\omega' = \emptyset$.
2. Otherwise, $\omega'$ is the connected component of good sites which contains 0.

Define

$$A = \{\omega : |\omega'| = \infty\}, \quad \theta(p,s) = \mathbf{P}_{p,s}[A].$$

If follows from the definition that $\theta(p,0) = \mathbf{P}_p[A_T]$ and $\theta(p,1) = \mathbf{P}_p[A_A]$.

Next, we approximate $\theta(p,s)$ by functions on finite volume spaces. For $v = (x,y) \in \mathbf{Z}^2$ let

$$B_v(k) = [x-k, x+k] \times [y-k, y+k],$$

and define:

$$A^N = \{\omega : \omega' \cap \partial B_0(N) \neq \emptyset\}, \quad \theta^N(p,s) = \mathbf{P}_{p,s}[A^N].$$

It is clear that for all $(p,s)$,

$$\theta(p,s) = \lim_{N\to\infty} \theta^N(p,s).$$

The core of the proof is the following comparison of the effect of the $p$ and $s$ percolation on the probability of the event $A^N$.

**Lemma 3.2** *There exists $N_0$ and a continuous positive function $g(p,s)$ on the square $(0,1)\times(0,1)$ such that for all $N \geq N_0$*

$$\frac{\partial \theta^N(p,s)}{\partial p} \leq g(p,s) \frac{\partial \theta^N(p,s)}{\partial s}.$$

**Proof of Theorem 1.6:** By Theorem 1.3 and Proposition 1.5 it follows that there exist $0 < q_0 < q_1 < 1$ and $0 < \bar{\epsilon} < 1$ such that

$$[q_1, 1] \subset (1-\bar{\epsilon}, 1] \subset \bar{W}(d) \subset [1-\bar{\epsilon}, 1] \subset [q_0, 1],$$

and

$$[q_1, 1] \subset W(d) \subset [q_0, 1],$$

where $\bar{W}(d)$ is the set of winning probabilities using a trivial strategy. Let $\alpha > 0$ be such that $[q_0, q_1 + \alpha] \subset (0,1)$. Let $g(p,s)$ be the continuous function that exists by Lemma 3.2, and

$$m = \max_{[q_0, q_1+\alpha]\times[1/4,3/4]} \{g(p,s), 1/\alpha\}.$$

Note that if $p \in [q_0, q_1]$, then $[p, p+\frac{1}{2m}] \subset [q_0, q_1+\alpha]$. By Lemma 3.2 we obtain for $p \in [q_0, q_1]$ and $N \geq N_0$:

$$\begin{aligned}
\theta^N(p, \frac{3}{4}) - \theta^N(p + \frac{1}{2m}, \frac{1}{4}) &= \int_0^{1/2} \frac{d\theta^N}{dt}(p + \frac{1}{2m} - \frac{t}{m}, \frac{1}{4} + t)dt \\
&= \int_0^{1/2} (\frac{\partial \theta^N}{\partial s} - \frac{1}{m}\frac{\partial \theta^N}{\partial p})(p + \frac{1}{2m} - \frac{t}{m}, \frac{1}{4} + t)dt \\
&\geq 0.
\end{aligned}$$

Note that $\theta^N(p,s)$ is a monotone function of $s$. Therefore when $p \in [q_0, q_1]$ and $N \geq N_0$:

$$\theta^N(p,1) \geq \theta^N(p, \frac{3}{4}) \geq \theta^N(p + \frac{1}{2m}, \frac{1}{4}) \geq \theta^N(p + \frac{1}{2m}, 0).$$

Taking the limit $N \to \infty$ we see that when $p \in [q_0, q_1]$, $\theta(p,1) \geq \theta(p + \frac{1}{2m}, 0)$. This implies that $(1-\epsilon, 1] \subset \{p : \mathbf{P}_p[A_A] > 0\} \subset W(d)$, where $\epsilon = \bar{\epsilon} + \frac{1}{2m} > \bar{\epsilon}$ as needed (note that the assumption $W(d) \subset [q_0, 1]$ implies in particular that $1 - \bar{\epsilon} - 1/2m \geq q_0$). $\square$

**Proof of Lemma 3.2:** Let $\gamma$ be a configuration inside the square $B_0(N)$ and $v$ a site. Let $\gamma^{p+}$ ($\gamma^{p-}$) be $\gamma$ where the site $v$, and this site only, is updated to be a free site (a mine). Call a site $v$ *p-pivotal* for the configuration $\gamma$, if $\gamma^{p+} \in A^N$ and $\gamma^{p-} \notin A^N$. Define a *s-pivotal* site similarly. By Russo's formula (see e.g. [5]):

$$\frac{\partial \theta^N(p,s)}{\partial p} \leq \mathbf{E}_{p,s}|\{v : v \text{ is } p\text{-pivotal for } A^N\}|$$

and

$$\frac{\partial \theta^N(p,s)}{\partial s} = \mathbf{E}_{p,s}|\{v : v \text{ is } s\text{-pivotal for } A^N\}|.$$

The proof will follow once we will show that there exists a positive continuous function $g(p,s)$ such that:

$$\mathbf{E}_{p,s}|\{v : v \text{ is } p\text{-pivotal for } A^N\}| \leq g(p,s)\mathbf{E}_{p,s}|\{v : v \text{ is } s\text{-pivotal for } A^N\}|. \tag{3}$$

Let $v$ be $p$-pivotal for a configuration $\gamma$. We will show that it is possible to change the configuration (of both the $p$-percolation and the $s$-percolation) inside the box $B_v(20)$, in order to obtain that one of the vertices in the box is $s$-pivotal. This implies (3). We assume below that $B_v(20) \subset B_N(0)$ and $d(v,0) \geq 10$. The cases where $v \notin B_{N-20}$ or $d(v,0) < 10$ are treated in a similar way.

Since $v$ is $p$ pivotal, it follows that $\gamma^{p-} \notin A^N$ and $\gamma^{p+} \in A^N$. Assume first that we may either set open all the $s$ percolation values of $\gamma^{p-}$ in $B_v(20)$ in order to obtain a configuration $\gamma^{p-s+} \in A^N$, or set closed all the $s$ percolation values of $\gamma^{p+}$ in $B_v(20)$ in order to obtain a configuration $\gamma^{p+s-} \notin A^N$. Clearly in either case, by changing a subset of these $s$-values we may obtain $\tilde{\gamma}$ such that for $\tilde{\gamma}$ there exists a $u \in B_v(20)$ which is $s$-pivotal for $A^N$. Since $\tilde{\gamma}$ was obtained from $\gamma$ by changing the configuration inside $B_v(20)$, the lemma follows.

If this assumption does not hold, then $\gamma^{p+s-} \in A^N$ and $\gamma^{p-s+} \notin A^N$. Since $\gamma^{p+s-} \in A^N$, there exists a self-avoiding path of good sites (in the configuration $\gamma^{p+s-}$) $\varphi = (0, v^1, v^2, \ldots, v^k = v, \ldots, v^n)$, such that $v^n \in \partial B_0(N)$.

By the definition of $\gamma^{p+s-}$ it follows that for all $i$, if $d_\infty(v^i, v) \leq 19$, then $v^i$ is a wide free site. Let

$$i = \min\{\ell < k : d_\infty(v^\ell, v) = 18\}, \quad j = \max\{\ell > k : d_\infty(v^\ell, v) = 18\}.$$

Note that all the neighbors of $v^i$ and $v^j$ are free sites. Also note that it cannot be the case that $v^i$ and $v^j$ are neighbors, as this will imply that $\gamma^{p-s+} \in A^N$. We need to consider several cases - where we analyze in detail the case which is illustrated in Figure 4. Here $v^i$ and $v^j$ belong to the same face of $B_v(18)$ and both $v^i$ and $v^j$ are at distance least 5 from the corners of $B_v(20)$. Since $\gamma^{p-s+} \notin A^N$, it follows that $d_\infty(v_i, v_j) \geq 4$. We let $v'$ be the neighbor of $v^i$ such that $v^i$ and $v'$ differ in exactly one coordinate, and $d_\infty(v', v) = 17$. Let $v''$ be a neighbor of $v^j$ which is defined similarly.

Let $\tilde{\gamma}$ be obtained from $\gamma^{p+s-}$ in the following way.

- For all $w$ with $d_\infty(v,w) \geq 18$, $\tilde{\gamma}$ has the same $p$ and $s$ percolation values as $\gamma^{p+s-}$.
- For all $w$ with $d_\infty(v,w) \leq 18$, $w$ is closed in the $s$ percolation.
- If $d_\infty(w,v) = 17$, then $w$ is a mine unless, $d_\infty(w,v') \leq 1$, or $d_\infty(w,v'') \leq 1$, in which case $w$ is a free site.
- $v'$ and $v''$ are connected by a path of almost wide free sites inside the box $B_{17}(v)$, in such a way that $v$ is the only mine which is adjacent to the path and it is $s$-pivotal for the existence of the path (Instead of giving a long explicit construction, we refer the reader to Figure 4).

We claim that $\tilde{\gamma}$ has $v$ as $s$-pivotal site. Let $\tilde{\gamma}^{s-} = \tilde{\gamma}$ (so that $v$ is closed in the $s$-percolation), and $\tilde{\gamma}^{s+}$ be $\tilde{\gamma}$ where the site $v$, and this site only, is updated as to be open for the $s$-percolation.

For $\tilde{\gamma}^{s+}$, the concatenation of $(v_0, \ldots, v^i, v')$, the path of good sites inside $B_{17}(v)$ connecting $v'$ to $v''$ and $(v'', v^j, \ldots, v^n)$ is a self avoiding path of good sites connecting $v^0$ to $\partial B_N(0)$. Thus, $\tilde{\gamma}^{s+} \in A^N$.

On the other hand, for $\tilde{\gamma}^{s-}$, the set of good sites $w$ with $d_\infty(v,w) \geq 17$ is a subset of the set of good sites with $d_\infty(v,w) \geq 17$ for the configuration $\gamma^{p+s-}$. Since $v$ is $p$-pivotal for $\gamma^{p+s-}$, it follows that for $\tilde{\gamma}$ there exists no path of good sites connecting 0 and $\partial B_N(0)$ which is disjoint from $B_{17}(0)$. For $\tilde{\gamma}^{-}$, there is no path of good sites connecting $u$ and $w$ with $d_\infty(u,v) = d_\infty(w,v) = 17$. Thus, $\tilde{\gamma}^{s-} \notin A^N$. It follows that $v$ is $s$-pivotal for $\tilde{\gamma}$ as needed.

It is easy to obtain similar constructions when $v^i$ and $v^j$ do not belong to the same face, or if either of $v^i$ or $v^j$ is close to a corner.

□.

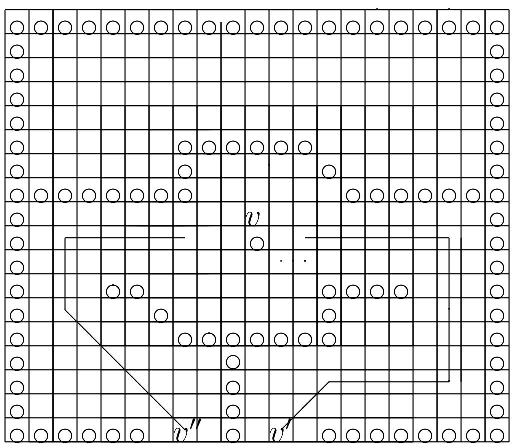


Figure 4: Making $v$ pivotal

We suspect that in some sense, a player who has access to better strategies should have a strictly larger set of winning probabilities. A *range $n$ strategy* is a strategy which decides to recover a site $v$ depending only on the information at distance at most $n$ from $v$. Let $W_n(d)$ be the set of probabilities for which one can win the game using range $n$ strategies. A general conjecture in the same spirit as Theorem 1.6 is:

**Conjecture 3.3** *There exists a $k$ such that for all $n$, there exists an $\epsilon = \epsilon(n) > 0$ such that*

$$(W_n(d) + (-\epsilon, \epsilon)) \cap [0, 1] \subset W_{n+k}(d).$$

# 4 Monotonicity and open problems

It may seem reasonable that the more free sites one has, the larger the probability is for winning the game. There are two natural interpretations of this conjecture.

**Interpretation 1:** If one can win the game for mines density $1-p$ for some graph $G$, one can also win it for density $1-p$ for any graph $\tilde{G}$ which has $G$ as a subgraph.

**Interpretation 2:** If one can win the game for a certain mines density $1-p$, he can also win it for any other mines density $1-\tilde{p}$ for $\tilde{p} > p$. That is, the set of winning probabilities is an interval $[p, 1]$ or $(p, 1]$.

We start with a motivation on how adding mines may increase the success probability. Look at the Figure 5. Note that by looking at the data we are given, we cannot decide between the 3 configurations on the left of the figure. Therefore given this data, we cannot even find one free vertex. On the other hand, by adding the 2 mines (on the right of the figure), the situation is clear and there are 3 surely free vertices.

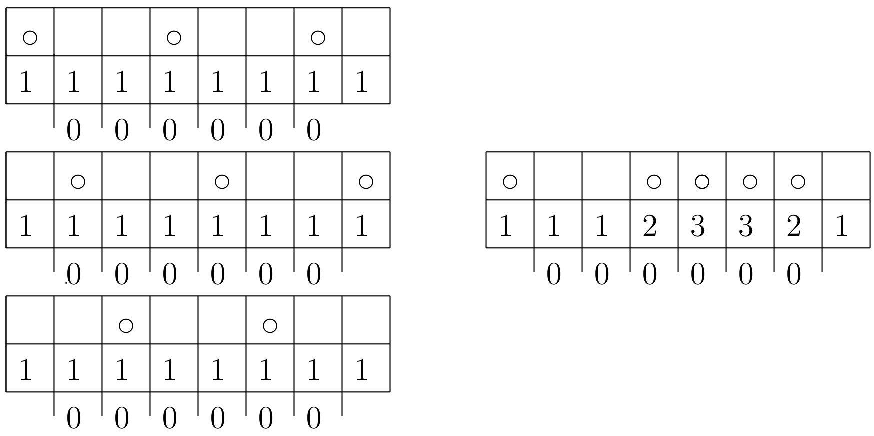


Figure 5: Adding mines may help

We now give an example in which our first interpretation of monotonicity does not hold, while the second one holds.

**Example 1:** Consider the minesweeper game on the rooted $(d+1)$-regular tree $T_d$. It is clear that we can win the game with positive probability if and only if there exists with positive probability an infinite sub-tree of vertices which are all free of mines and in which all the vertices have either no children or $d$ children.

Thus we are dealing with a branching process (see e.g. [3]) in which the probability that any vertex survives is $p^d$. Therefore there is a positive probability of winning the game iff

$$dp^d > 1.$$

Therefore, the set of winning probabilities for the $d+1$ regular tree, denoted $W(d)$, is the interval $((1/d)^{1/d}, 1]$. Note that $\{T_d\}_{d=3,\infty}$ is an increasing family of graphs, but $\{W(d)\}_{d=3,\infty}$ is decreasing and has as intersection the set $\{1\}$.

In the second example we give a variant of the game for which the first interpretation of monotonicity does not hold.

**Example 2:** This is a variation on the last example. Suppose that instead of the number of adjacent vertices occupied by mines we are given the following information. For every vertex we are given one bit of information. This bit is on iff there are no mines in the left-most $l$ children and there is at least one mine in $d-l$ other children. Again, this is reduced to a branching process. But here the expected number of children is:

$$lp^l(1-p^{d-l})$$

For every value of $l$, when $d$ is large, the set $W(d)$ is some open interval $(p_1, p_2)$.

For another example of nonmonotic behavior in percolation-related models we refer the reader to [4].

We conclude with one open problem and one conjecture.

**Conjecture:** *On $\mathbf{Z}^d$ the mine sweeper game is monotone in $p$, i.e., the set of winning probabilities is an interval $[p(d), 1]$ or $(p(d), 1]$.*

**Problem:** *What can be said about the set $W(d)$ for $\mathbf{Z}^d$ as $d \to \infty$ ?*

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